Additional Notes for Expected Values of Joint Distributions

This is an addition to section 29.3.

In section 29.3, the examples only include the expected values of sums and differences of two continuous random variables (Remark 29.14) and using one variable (Remark 29.13) even though the generalized function of $\mathbb{E}\big(g(x,y)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dydx$ is provided in Definition 29.12. I wanted to provide another example for the generalized remark which involves the expected value of another function of both x and y.

Example: Let X and Y have the joint density:

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{5}(x^2+y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & else \end{cases}$$

Find $\mathbb{E}$(XY).

Answer:
According to Definition 29.12,

$$\mathbb{E}\big(g(x,y)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dydx$$

Therefore

$$\mathbb{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dydx = \int_0^1\int_0^1 xy\frac{6}{5}(x^2+y)dydx = \frac{6}{5}\int_0^1\int_0^1 (x^3y+xy^2)dydx$$

$$= \frac{6}{5}\int_0^1 \left(\frac{x^3y^2}{2}+\frac{xy^3}{3}\right)\Bigg|_0^1 dx = \frac{6}{5}\int_0^1\left(\frac{x^3}{2}+\frac{x}{3}\right)dx = \frac{6}{5}\left(\frac{x^4}{4\cdot 2}+\frac{x^2}{2\cdot 3}\right)\Bigg|_0^1 = \frac{6}{5}\left(\frac{1}{8}+\frac{1}{6}\right) = \frac{7}{20}$$

$$= 0.35$$

Just remember, that you can calculate the expected value of any function. If you are interested in why I chose $\mathbb{E}(XY)$, this expected value is used in the calculation of the covariance which is the first step in theoretically calculating the correlation coefficient (Chapter 39).